Synchronization limit of weakly forced nonlinear oscillators

2014 J. Phys. A: Math. Theor. 47 402002

(http://iopscience.iop.org/1751-8121/47/40/402002)

OPEN ACCESS
IOP Publishing

Fast Track Communications
Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **47** (2014) 402002 (10pp) doi:10.1088/1751-8113/47/40/402002

Fast Track Communication

Synchronization limit of weakly forced nonlinear oscillators

Hisa-Aki Tanaka

Graduate School of Informatics and Engineering, The University of Electro-Communications, Chofu, Tokyo 182–8585, Japan

E-mail: htanaka@uec.ac.jp

Received 13 May 2014, revised 22 July 2014
Accepted for publication 25 July 2014
Published 23 September 2014

Abstract

Nonlinear oscillators exhibit synchronization (injection-locking) to external periodic forcings, which underlies the mutual synchronization in networks of nonlinear oscillators. Despite its history of synchronization and the practical importance of injection-locking to date, there are many important open problems of an efficient injection-locking for a given oscillator. In this work, I elucidate a hidden mechanism governing the synchronization limit under weak forcings, which is related to a widely known inequality; Hölder's inequality. This mechanism enables us to understand how and why the efficient injection-locking is realized; a general theory of synchronization limit is constructed where the maximization of the synchronization range or the stability of synchronization for general forcings including pulse trains, and a fundamental limit of general $m:n$ phase locking, are clarified systematically. These synchronization limits and their utility are systematically verified in the Hodgkin–Huxley neuron model as an example.

S Online supplementary data available from stacks.iop.org/jpa/47/402002/mmedia

Keywords: synchronization, entrainment, injection-locking, fundamental limit, Arnold tongue, optimization, Hölder's inequality
PACS numbers: 05.45.Xt, 05.45.-a, 84.35.+i

(Some figures may appear in colour only in the online journal)

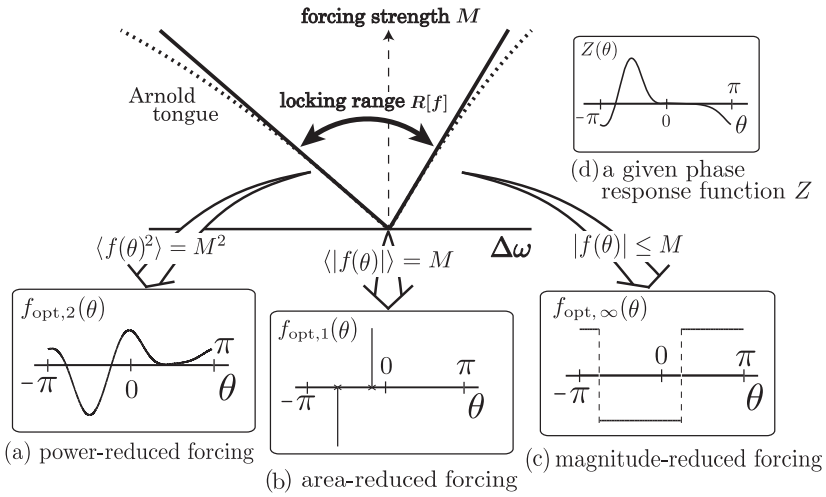

Figure 1. Best forcings revealed by the mechanism of synchronization limits when maximizing the range of entrainment: (a) power-reduced forcing in **P1**, (b) area-reduced forcing in **P2**, and (c) magnitude-reduced forcing in **P2**, obtained for (d) a given phase response function Z (taken from the Hodgkin–Huxley phase model [17] as an example). $f_{\mathrm{opt},\,p}(p = 1, 2, \text{and } \infty)$ here is determined algorithmically in the main text.

1. Introduction

Entrainment, which adjusts the frequencies of oscillators to that of an external forcing (signal) above a critical forcing amplitude, is a fundamental phenomenon of wide interest with a long history and a large variety of applications [1–5]. Generally, the ratio of these two frequencies is locked to $m:n$ in the entrainment, which is called $m:n$ frequency locking. Likewise, if the oscillation phase of the oscillator $(=\theta)$ and that of the external forcing $(=\theta_{\mathrm{ext}})$ satisfy $n\theta - m\theta_{\mathrm{ext}} = \text{const}$, it is called $m:n$ phase locking. And, $1:1$ phase locking is termed synchronization [4]. As opposed to conventional single-oscillator entrainment, entrainment has significance even in coupled oscillatory elements, thanks to recent studies on networks of coupled oscillators [6–9]. Simultaneously, in many branches of science and engineering, engineering entrainment (injection-locking) with an efficient forcing (injection signal) has become more important, and methods for efficient entrainment have been developed in recent years [10–17], reflecting advances in experimental techniques for observing and manipulating such oscillators (see [16–20] for instance).

Despite its history, and the advances made in and the wide utility of synchronization to date, it has been open problems of practical and theoretical significance to realize the synchronization limit. Part of our results realizing the synchronization limit is schematically illustrated as in figure 1, which solve the three basic open problems, posed in **P1**, **P2**, and **P3** below.

P1: What sort of mechanism determines the best (i.e., global optimal) power-reduced (average square of its waveform is bounded) forcing that produces the maximum entrainment range or the most stable synchronization? And are the optimal power-reduced forcings obtained in [16, 17] really the best ones if a more general class (function space) of forcings is considered?

P2: In conjunction with **P1**, what determines the best area-reduced (total absolute value of its waveform is bounded) forcing or magnitude-reduced (its maximum amplitude is bounded) forcing?

P3: Do the answers to **P1** and **P2** regarding $1:1$ phase locking remain valid for general $m:n$ phase locking? If so, how is the best forcing for $m:n$ phase locking related to that for $1:1$ phase locking? Or, if not, what limits an ideal (most efficient) forcing for $m:n$ phase locking?

Recent studies have approached these problems, from different formulations using calculus of variations ([16] for optimizing the locking range, [17] for optimizing the stability of phase locking, and [21] for both of them in $m:n$ phase locking, for a power-reduced case). However, regarding **P1**, these conventional studies only suggest the existence of a synchronization limit by finding (possibly local optimal) forcings in each particular situation, and regarding **P2** and **P3**, methods used in these conventional studies are not applicable, and hence a fundamental limit of entrainability has not been clarified. Toward this end, in this paper, we find an underlying mechanism in the above three problems leads us to a unified, global view of synchronization limits and their constructions.

2. Entrainment modeled by the phase equation

Here we introduce the well-known phase equation for the weakly forced nonlinear oscillators [1–5]. The entrainment process of a limit-cycle oscillator in the weak forcing limit can be modeled by $\dot{\psi} = \omega + \epsilon Z(\psi) f(\Omega t)$, where ψ is the phase variable of the oscillator ($\psi \in [-\pi, \pi] \equiv S$), Z is the phase response (sensitivity) function, and ω and Ω are the natural frequency of the oscillator and the frequency of the weak forcing $\epsilon f(\Omega t)$, respectively, following the notation in [2]. In general, $m:n$ phase locking occurs when $\frac{\omega}{m} \approx \frac{\Omega}{n}$ is satisfied for positive relatively prime integers m and n. In this situation, the above equation is further simplified by the method of averaging (after setting ϵ to 1) to the following phase equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Delta\omega + \Gamma_{m/n}(\phi), \tag{1}$$

where ϕ and $\Delta\omega$ satisfy $\phi = \psi - \frac{m}{n}\Omega t$ and $\Delta\omega = \omega - \frac{m}{n}\Omega$, respectively, and the interaction function $\Gamma_{m/n}(\phi)$ is determined by f and Z as $\Gamma_{m/n}(\phi) = \frac{1}{T}\int_0^T Z\left(\frac{m}{n}\Omega t + \phi\right) f(\Omega t)\,\mathrm{d}t \equiv \frac{1}{2\pi}\langle Z(m\theta + \phi) f(n\theta)\rangle$, in which $T = \frac{2\pi n}{\Omega}$ (n times the natural period of the oscillator), $\theta \in [-\pi, \pi]$ represents $\frac{\Omega t}{n}$, and $\langle\,\cdot\,\rangle$ denotes the integration over its period 2π: $\langle\,\cdot\,\rangle \equiv \int_{-\pi}^{\pi} \cdot\ \mathrm{d}\theta$. When considering the case of $m = n = 1$, i.e., $1:1$ entrainment, we will abbreviate $\Gamma_{m/n}$ as Γ, for simplicity.

Now, we define the synchronizability of the oscillator by equation (1), which is composed of the forcing f and the phase sensitivity Z, and we introduce some practical constraints on f and Z.

We first consider a general class of periodic functions $f(\theta)$ as the weak forcing, namely, those satisfying the following constraints

$$\|f\|_p \equiv \left\langle |f(\theta)|^p \right\rangle^{\frac{1}{p}} = M \quad \text{and} \quad \frac{1}{2\pi}\langle f(\theta)\rangle = 0, \tag{2}$$

in which both p and M are positive constants; here we assume $f \in L^p(S)$, namely, that f is an L^p-function on $S \equiv [-\pi, \pi]$. Henceforth, we assume $p \geqslant 1$, due to the following physical interpretation of the constraints (2). First, for $p = 2$, $\|f\|_p = M$ becomes $\langle f^2\rangle = M^2$, i.e., the

average square of f (the power) is fixed at M^2, which is the case considered in [16, 17, 21], and this case corresponds to the power-reduced forcing in **P1**[1]. (An example is shown in figure 1(a).) For $p=1$, $\|f\|_p = M$ corresponds to the area-reduced forcing in **P2**: $\langle |f(\theta)| \rangle = M$. (See figure 1(b).). On the other hand, the case of $p = \infty$ implies the magnitude-reduced forcing in **P2** (figure 1(c)), because $\|f\|_\infty = M$: $|f(\theta)| \leqslant M$ for almost every $\theta \in S$, since $\|f\|_\infty$ is the essential supremum of $|f(\theta)|$. Thus, the constraints (2) continuously cover various situations in a natural way. Besides $\|f\|_p = M$, another constraint $\frac{1}{2\pi}\langle f(\theta) \rangle = 0$ in equation (2), i.e., a charge-balance constraint [22, 23], is introduced, because it is required in practical situations where total injection (injected charge) should be 0.

On the other hand, for the phase sensitivity $Z(\theta)$, we assume a general class of Z being twice differentiable for the case of $1 < p \leqslant \infty$, and Z being locally Lipschitz continuous for $p = 1$, which is required to prove existence of the optimal forcing; detailed information is given in appendix A of the supplementary material (available at stacks.iop.org/jpa/47/402002/mmedia). We note that these (mild) assumptions are normally satisfied for the oscillators in real environments.

Phase locking occurs when the phase difference is locked, i.e., $\mathrm{d}\phi/\mathrm{d}t = \Delta\omega + \Gamma(\phi) = 0$. The range of frequency difference $\Delta\omega$, where the solution for a stable steady state exists for ϕ, defines the locking range $R[f]$ for a certain fixed forcing waveform [4]. Therefore, the locking range is the difference between the maximum (at $\phi = \phi_+$) and minimum (at $\phi = \phi_-$) values of $\Gamma(\phi)$ where the phase locking is maintained. $R[f]$ is thus given by $\Gamma(\phi_+) - \Gamma(\phi_-) = \frac{1}{2\pi}\langle f(\theta)[Z(\theta+\phi_+) - Z(\theta+\phi_-)] \rangle$ [16].

3. Fundamental limits of synchronization

Now, we formulate the optimal synchronization problem: the global optimal forcing waveform maximizes the locking range $R[f]$ under the constraints (2), which gives the maximum width of the Arnold tongue for the weak forcing, as in figure 1. If we assume f to satisfy $\|f\|_p = M$, then for maximizing $R[f]$ under the constraint $\frac{1}{2\pi}\langle f(\theta) \rangle = 0$, the functional $J[f] \equiv R[f] + \frac{\lambda}{2\pi}\langle f(\theta) \rangle$ is introduced, where λ is a Lagrange multiplier. Moreover, $J[f]$ is rewritten as the following inner product of f and g

$$J[f] = \frac{1}{2\pi}\left\langle f(\theta)\left[\bar{Z}(\theta) + \lambda\right]\right\rangle \equiv \frac{1}{2\pi}\langle f(\theta) g(\theta) \rangle, \tag{3}$$

where $g(\theta) = \bar{Z}(\theta) + \lambda$, $\bar{Z}(\theta) \equiv Z(\theta + \Delta\phi) - Z(\theta)$ and $\Delta\phi \equiv \phi_+ - \phi_-$, after moving to the new coordinate: $\theta + \phi_- \to \theta$. On the other hand, when maximizing the (linear) stability of the phase locking, $R[f]$ is simply replaced with $S[f] \equiv -\frac{1}{2\pi}\langle f(\theta) Z'(\theta) \rangle$ ($\approx -\Gamma'(\phi_*)$ where ϕ_* denotes a stable fixed point for equation (1). Note this ϕ_* is set to 0 on the new coordinate.) if $\Delta\omega = 0$ in (1) [17]. Thus, for the case of maximizing the stability $S[f]$, the same arguments for maximizing the locking range $R[f]$ are possible simply by replacing $g(\theta) = \bar{Z}(\theta) + \lambda$ to $g(\theta) = -Z'(\theta)$ in equation (3). Henceforth, we restrict our argument to the case of maximizing locking-range in the proof below, for the sake of simplicity. For the physical significance of $R[f]$ and $S[f]$, see [16, 17] respectively.

The optimization of $J[f]$ is identical to the optimization of $R[f]$ under the constraints (2), and hereafter we denote the global optimal forcing for a given p as $f_{\mathrm{opt},\,p}$. For this optimization problem, local optimal forcings can be captured by using the calculus of variations,

[1] Besides the case for $p = 2$, the case for $p > 2$ can be useful for engineering nonlinear systems with generalized energy functions.

such as the Euler–Lagrange equation for $p > 1$ (or the bang–bang principle [24] for $p = \infty$). But, this approach has a limitation: it is intrinsically local and heuristic, and its result lacks global information. Namely, we cannot understand 'how' and 'when' the global optimal forcing is realized. Furthermore, it is impossible using this approach to show that a certain arbitrarily tall pair of pulses realize the entrainment limit for $p = 1$ in **P2**, as explained later. Hence, by using only the calculus of variations, it is hard or impossible to answer questions **P1**, **P2**, and **P3** regarding the physical limit of synchronizability.

However, if we realize that equation (3) with the constraint $\|f\|_p = M$ corresponds to Hölder's inequality, $\|fg\|_1 \leqslant \|f\|_p \|g\|_q$ in which $1 \leqslant p,\ q \leqslant \infty$ and $p^{-1} + q^{-1} = 1$, then answers to the basic questions **P1**, **P2**, and **P3** are systematically obtained as *theorems* and their proofs are obtained, as follows.

3.1. 1 : 1 *phase locking for* $1 < p < \infty$

First, regarding **P1** for $1 < p < \infty$, the question is answered by direct use of the *equality condition* of Hölder's inequality: $\| fg \|_1 = \| f \|_p \| g \|_q$ holds if and only if there exist constants r and s, not both 0, such that $r|f(\theta)|^p = s|g(\theta)|^q$ for almost all $\theta \in S$ [26]. Having this equality condition in mind, a candidate of the optimal f_* for $J[f]$ under the constraints (2) is given as $f_*(\theta) = M\sigma(\theta)(|g(\theta)|/\| g \|_q)^{\frac{1}{p-1}}$ with $\sigma(\theta)$ being any function having either ± 1 values for $\theta \in S$, by assuming the second equality holds in the following general relationship: $2\pi J[f] = \langle fg \rangle \leqslant \langle | fg | \rangle = \| fg \|_1 \leqslant \| f \|_p \| g \|_q = M \| g \|_q$. Further, by assuming the first equality in the above relationship, we can narrow the above f_* uniquely to

$$f_*(\theta) = M \ \mathrm{sgn}\,[g(\theta)] \left(|g(\theta)|/\| g \|_q\right)^{\frac{1}{p-1}}, \tag{4}$$

where sgn is the signum function defined as $\mathrm{sgn}\,(x) = -1$ (for $x < 0$), 1 (for $x > 0$), 0 (for $x = 0$).

Here we call this f_* as an *ideal* forcing since it realizes a possible *ideal* entrainment of the maximum locking range, and we now assume such an f_* to exist (which is later verified from equation (5)). Under this assumption, for any given Z, $\langle fg \rangle$ $(= 2\pi J[f])$ is given as $\langle f_* g \rangle = \| f_* \|_p \| g \|_q = M \langle | \bar{Z}(\theta) + \lambda |^q \rangle^{\frac{1}{q}}$ $(= 2\pi J[f_*])$. Then, the ideal locking range $J[f_*]$ is a function of $\Delta\phi$ and λ, for a given Z and p. In order to maximize $J[f_*]$, the function $\left\langle |\bar{Z}(\theta) + \lambda|^q \right\rangle$ should be maximized by tuning the two parameters $\Delta\phi$ and λ under the constraints (2). For this purpose, we define the following functions: $F(\Delta\phi, \lambda) \equiv \left\langle |\bar{Z}(\theta) + \lambda|^q \right\rangle$ and $G(\Delta\phi, \lambda) \equiv \langle\, \mathrm{sgn}\left[\bar{Z}(\theta) + \lambda\right] | \bar{Z}(\theta) + \lambda |^{\frac{1}{p-1}} \rangle$. Then $G(\Delta\phi, \lambda) = 0$ is obtained from the constraint $\langle f(\theta) \rangle = 0$ after plugging equation (4) into it. Finally, for maximizing $F(\Delta\phi, \lambda)$ under the constraint $G(\Delta\phi, \lambda) = 0$, the function $H(\Delta\phi, \lambda) \equiv F(\Delta\phi, \lambda) + \mu G(\Delta\phi, \lambda)$ is introduced, where μ is a Lagrange multiplier. Thus, the optimal entrainment problem is reduced down to the finite-dimensional optimization of $H(\Delta\phi, \lambda)$, and the above argument clarifies the mechanism of how optimal forcings are realized.

Straightforward calculations show that the optimal solutions $(\Delta\phi_*, \lambda_*)$ to $H(\Delta\phi, \lambda)$ are determined from the following equations

$$\partial H/\partial \Delta\phi = \alpha \left\langle \mathrm{sgn}\,[g(\theta)]\ |g(\theta)|^{\frac{1}{p-1}} Z'(\theta + \Delta\phi) \right\rangle = 0, \tag{5a}$$

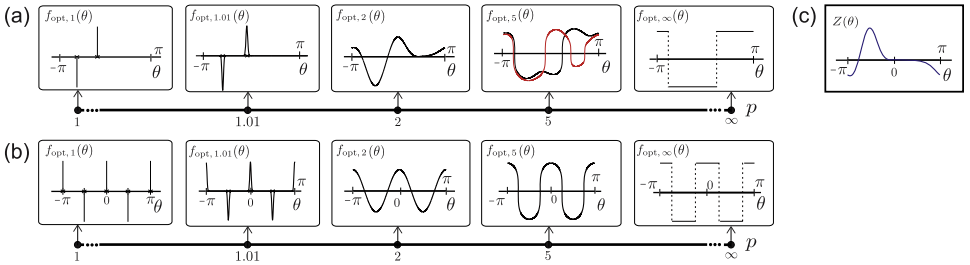

Figure 2. Overview of all optimal forcings for various $p \in [1, \infty]$ obtained for the Hodgkin–Huxley (HH) neuron phase model [17]. Black, red, and blue curves represent the global optimal, the second optimal, and Z, respectively. Panel (a), (b), respectively, show $1:1$ and $1:2$ phase locking optimal forcings for the HH neuron model [17], having the associated $Z(\theta) = 0.176116 + 0.371736\cos\theta - 0.740283\sin\theta - 0.819478\cos 2\theta + 0.00225226\sin 2\theta + 0.181875\cos 3\theta + 0.403816\sin 3\theta + 0.111446\cos 4\theta - 0.0892503\sin 4\theta - 0.0127103\cos 5\theta - 0.0165083\sin 5\theta$ as shown in figure 2(c).

$$\partial H/\partial\lambda = \alpha\left\langle \mathrm{sgn}\left[g(\theta)\right]\,|g(\theta)|^{\frac{1}{p-1}}\right\rangle = 0, \tag{5b}$$

$$|\mathcal{H}(H)| = \mathcal{H}_{13}\left(\alpha\mathcal{H}_{12}^2 - \mathcal{H}_{13}\mathcal{H}_{22}\right) > 0, \tag{5c}$$

where $\alpha = \frac{p}{p-1}$, $g(\theta) = \bar{Z}(\theta) + \lambda$, $1 < p < \infty$, and $\mathcal{H}(H)$ represents the bordered Hessian matrix of H; detailed information on $\mathcal{H}(H)$ is given in appendix B of the supplementary material (available at stacks.iop.org/jpa/47/402002/mmedia). Note, for every $(\Delta\phi_*, \lambda_*)$, the associated f_* indeed maximizes the associated Γ at ϕ_+ and minimizes Γ at ϕ_- by straightforward calculation, although its detail is omitted here.

To determine $(\Delta\phi_*, \lambda_*)$ of the ideal forcing f_* in equation (4), we have numerically solved equations (5a) and (5b), and checked whether the obtained $(\Delta\phi, \lambda)$ satisfies equation (5c); the results for the example in figure 2 are listed in appendix C of the supplementary material (available at stacks.iop.org/jpa/47/402002/mmedia). Since all possible $(\Delta\phi_*, \lambda_*)$ can be obtained numerically, and from the above argument concerning the equality condition and the associated equation (4), all global or local optimal forcings are captured in $L^p(s)$ [2]. Among them, the global optimal (i.e., the best) forcing $f_{\mathrm{opt},\,p}$ is identified, simply by comparing the associated locking ranges $R[f]$, as shown in figures 2(a) and (b) for $p = 1,\ 1.01,\ 2,\ 5,\ \infty$ for $Z(\theta)$ shown in figure 2(c), respectively. The above steps constitute the algorithm for realizing the global optimal forcing, i.e., the fundamental limit of injection-locking[3]. Thus, the answer to **P1** has been obtained.

[2] These optimal forcings are obtained in $L^p(S)$. This implies that these optimals belong to a broader class of functions, compared with the one considered in the calculus of variations. More precisely, in the calculus of variations, we usually assume the function space to consist of piecewise-smooth functions (or absolutely continuous functions, at best), which is a subset of L^p. Namely, the result here is stronger than the one obtained by the conventional calculus of variations.

[3] Note, for the special case of $p = 2$, equation (5b) gives $\lambda = 0$, equation (5a) becomes $\langle\bar{Z}(\theta)Z'(\theta + \Delta\phi)\rangle = 0$, equation (5c) becomes $\langle\bar{Z}(\theta)Z''(\theta + \Delta\phi)\rangle < 0$, and hence the result for the power-reduced forcing in [16] is naturally recovered. This implies the optimal forcing obtained in [16] (as well as in [17]) is nothing other than the global optimal forcing in the more general function space of forcings, i.e., $L^2(S)$ (cf footnote 2).

3.2. 1 : 1 *phase locking for p = 1 and p = ∞*

Next, regarding **P2**, the question is answered by utilizing the cases of $p = 1$ and $p = \infty$ in Hölder's inequality. This can be done in a rigorous way [25], but due to space limitations, we here present a more intuitive explanation; taking the limits $p \to \infty$ and $p \to 1$ in equation (4), results respectively in

$$f_*(\theta) \to M \ \mathrm{sgn}\,[g(\theta)], \quad \text{pointwise for any } \theta \in S, \tag{6a}$$

$$f_*(\theta) \to \begin{cases} 0, & \text{pointwise for } \theta \neq \theta_*, \\ \infty, & \text{for } \theta = \theta_* \text{ with } g(\theta_*) > 0, \\ -\infty, & \text{for } \theta = \theta_* \text{ with } g(\theta_*) < 0, \end{cases} \tag{6b}$$

where θ_* represents a maximal point of $|g(\theta)|$ in S. A derivation of equations (6) is given in appendix D of the supplementary material (available at stacks.iop.org/jpa/47/402002/mmedia). In fact, these two limits are, respectively, consistent with the optimal forcings $f_{\mathrm{opt},\,\infty} = M \ \mathrm{sgn}\,[g(\theta)]$ which satisfies the bang–bang principle [24] and $f_{\mathrm{opt},\,1}$ which is a pair consisting of one arbitrarily tall negative pulse and one arbitrarily tall positive pulse (i.e., bipolar pulses) separated by $\Delta\phi_{\max}$ [4], as proved in appendix A.2 of the supplementary material (available at stacks.iop.org/jpa/47/402002/mmedia). Note this $\Delta\phi_{\max}$ is determined algorithmically from Z, as shown in appendix A.2 of the supplementary material (available at stacks.iop.org/jpa/47/402002/mmedia). Thus, the answer for the case $p = 1,\ \infty$ in **P2** is obtained, and the resulting best forcings are shown in figure 2(a), respectively, for $p = 1,\ \infty$. For the practical significance of such pulse forcings ($p = 1$) as well as square forcings ($p = \infty$), see [14, 15, 18, 23] for instance.

Now, we can design efficient injection-locking toward the synchronization limits. Such a design example is given in appendix E of the supplementary material (available at stacks.iop.org/jpa/47/402002/mmedia) for the HH neuron model [17].

3.3. *General m : n phase locking*

Finally, regarding **P3**, for simplicity, here we assume that Z, f, and the associated Z_n and f_m are given as Fourier series: $Z(\theta) = \frac{a_0}{2} + \sum_{\alpha\in\mathbb{N}_1}(a_\alpha \cos\alpha\theta + b_\alpha \sin\alpha\theta)$, $f(\theta) = \frac{c_0}{2} + \sum_{\alpha\in\mathbb{N}_1}(c_\alpha \cos\alpha\theta + d_\alpha \sin\alpha\theta)$, $Z_n(\theta) \equiv \frac{a_0}{2} + \sum_{\alpha\in n\mathbb{N}_1}(a_\alpha \cos\alpha\theta + b_\alpha \sin\alpha\theta)$, and $f_m(\theta) \equiv \frac{c_0}{2} + \sum_{\alpha\in m\mathbb{N}_1}(c_\alpha \cos\alpha\theta + d_\alpha \sin\alpha\theta)$, in which $\mathbb{N}_1$ is the natural numbers $=\{1, 2, ...\}$, $n\mathbb{N}_1$ denotes the multiples of $n = \{n, 2n, \cdots\}$, and $m\mathbb{N}_1$ denotes the multiples of m. Namely, Z_n and f_m are respectively the partial sum of the Fourier series Z and f. Also, we assume the resulting $\Gamma_{m/n}(\phi) = \frac{1}{2\pi}\langle Z(m\theta + \phi) f(n\theta)\rangle$ to be obtained by integrating term by term from the Fourier series of Z and f [5]. Then, from a trigonometric identity, we obtain $\Gamma_{m/n}(\phi) = \frac{1}{2\pi}\langle Z_n(m\theta + \phi) f(n\theta)\rangle = \frac{1}{2\pi}\langle Z_n(m\theta + \phi) f_m(n\theta)\rangle$. This implies that for general $m:n$ phase locking, the global optimal forcing $f_{\mathrm{opt},\,p}$ is obtained in the form of the above $f_m(n\theta)$ simply by replacing Z by Z_n in the optimization algorithms related to **P1** and **P2**. Note,

[4] The reason we represent $f_{\mathrm{opt},\,1}$ in such an asymptotic form (rather than using a formal delta function) is that $f_{\mathrm{opt},\,1}$ belongs to $L^1(S)$ from the context of Hölder's inequality, and that what counts here for optimization is the resulting $\Gamma_0(\phi)$ (rather than the form of $f_{\mathrm{opt},\,1}$ itself).

[5] Since we have assumed Z is twice differentiable for $1 < p \leqslant \infty$, here we further assume $f\,[\in L^p(S)]$ is piecewise continuous, which implies that $\langle Zf\rangle$ is piecewise smooth and it is obtained by integrating term by term.

there is only one exceptional case: $m:1$ phase locking, where an 'asymptotically' best forcing is constructed[6]. Thus, the answer to **P3** has been obtained.

Figure 2(b) shows the global optimal $1:2$ phase locking forcings of the Hodgkin–Huxley (HH) neuron phase model [17] for various p. We observe that these forcings have simpler waveforms (compared with the $1:1$ cases in figure 2(a)), uniformly spaced bipolar pulses ($p=1$), nearly sinusoidal ($p=2$), and almost uniformly spaced rectangles ($p=\infty$), which is a typical feature for $1:n$ global optimal forcing. The reason is as follows. In $1:n$ phase locking, the above $Z_n(\theta)$ determines the optimal forcings (through the algorithms related to **P1** and **P2**); only the multiples of n-th harmonics in the original $Z(\theta)$ affect the optimal forcings. In addition, as we observed the Fourier coefficients of Z in figure 2 caption, the second ($n=2$) harmonic dominates other higher harmonics, resulting in a virtually sinusoidal oscillation (analogous to the one near the Hopf bifurcation point). This situation becomes more typical if we consider a larger n, since the magnitude of higher multiples of n-th harmonics in $Z(\theta)$ decays sufficiently fast as the multiples of n becomes large for generic limit-cycle oscillators. In fact, this insight explains the reason for the more sinusoidal-like best forcings systematically obtained for a larger n in the general $m:n$ case (under a different but similar setting) [21].

4. Conclusion and discussion

In conclusion, we have proved a mechanism governing synchronization (injection-locking) limits, and clarified how and why the best forcing realizes the synchronization limit, by unveiling a hidden aspect, i.e., Hölder's inequality, behind it, for a general class of externally forced limit-cycle oscillators. Namely, synchronization limit is now characterized as the *equality condition* of Hölder's inequality. To the best of the author's knowledge, no previous study has addressed this mechanism or the existence of the synchronization limit. Since the phase equation (1) appears in many areas of science and engineering, the obtained results here have direct, broad impacts, as follows.

(i) Designing the best forcing (beyond power-reduced forcings, including pulse trains) and (simultaneously) designing a better phase sensitivity (i.e., oscillators with better entrainability) is one of the direct consequences, since optimization of the forcing f and that of Z are now equivalent by permuting Z and f in equation (3). Such an application is feasible for a practical system in real environments, for instance as in [27, 28]. Also, it is noted that our results for the area-reduced forcings here enables us to design the optimal pulse trains for injection-locking for the first time.

(ii) In addition, designing an efficient 'coupling' between oscillators for better mutual synchronization [29] is also promising, since the results of a single oscillator with one external forcing here is modified to the case of mutual synchronization straightforwardly.

(iii) Furthermore, the algorithm for realizing the optimality here should provide a unified, systematic method for optimally entraining a given oscillation pattern in an ensemble of oscillators (or excitable elements) with global coupling [6], as well as with local coupling

[6] The construction is as follow. Starting from m copies with the optimal forcing with prime period T_0 for $1:1$ phase locking, add a certain small perturbation such that the m copies of the forcing become a single forcing with prime period mT_0 while still satisfying the constraints (2). The resulting locking range becomes arbitrarily close to the ideal one (which is realized only in $1:1$ phase locking) as the perturbation becomes smaller, since the associated $\Gamma_{m/1}$ in equation (1) becomes arbitrarily close to the $\Gamma_{1/1}$ of the best forcing for $1:1$ phase locking.

[8], since this oscillation pattern is regarded as a limit-cycle oscillation in higher phase space, which is described by equation (1).

(iv) Though our present framework focuses on the noiseless case, noisy oscillators can be treated in the same way as here, by virtue of the recent progress in this direction [30, 31]. These applications and extensions (i)–(iv) of our findings are of high importance for further theoretical as well as experimental work, which will be reported in the near future.

Acknowledgments

The author (HT) is indebted to Dr Y Tsubo and Dr T Shimada for their enlightening suggestions and critical reading of the manuscript. H T is also grateful to the anonymous referees for their critical comments. This work was supported by MEXT (No. 23360047) and by the Support Center for Advanced Telecommunications Technology Research (SCAT).

References

[1] Winfree A T and Rethwisch D G 2001 *The Geometry of Biological Time* 2rd edn (New York: Springer)
[2] Kuramoto Y 1984 *Chemical Oscillations, Waves and Turbulence* (Berlin: Springer)
[3] Pikovsky A S, Rosenblum M G and Kurths J 2001 *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge: Cambridge University Press)
[4] Hoppensteadt F C and Izhikevich E M 1997 *Weakly Connected Neural Networks* (New York: Springer)
[5] Ermentrout G B and Terman D H 2010 *Mathematical Foundations of Neuroscience* (New York: Springer)
[6] Kori H, Kawamura Y, Nakao H, Arai K and Kuramoto Y 2009 *Phys. Rev.* E **80** 036207
[7] Kawamura Y, Nakao H and Kuramoto Y 2011 *Phys. Rev.* E **84** 046211
[8] Nakao H, Yanagita T and Kawamura Y 2012 *Procedia IUTAM* **5** 227–33
[9] Sato M, Hubbard B E, Sievers A J, Ilic B, Czaplewski D A and Craighead H G 2003 *Phys. Rev. Lett.* **90** 044102
[10] Feng J and Tuckwell H C 2003 *Phys. Rev. Lett.* **91** 018101
[11] Forger D and Paydarfar D 2004 *J. Theor. Biol.* **230** 521–32
[12] Lebiedz D, Sager S, Bock H G and Lebiedz P 2005 *Phys. Rev. Lett.* **95** 108303
[13] Gintautas V and Hübler A W 2008 *Chaos* **18** 033118
[14] Bagheri N, Stelling J and Doyle F J 2008 *PLoS Comput. Biol.* **4** e1000104
[15] Gat O and Kielpinski D 2013 *New J. Phys.* **15** 033040
[16] Harada T, Tanaka H A, Hankins M J and Kiss I Z 2010 *Phys. Rev. Lett.* **105** 088301
[17] Zlotnik A, Chen Y, Kiss I Z, Tanaka H A and Li J S 2013 *Phys. Rev. Lett.* **111** 024102
[18] Takano K, Motoyoshi M and Fujishima M 2007 *Proc. IEEE Asian Solid-State Circuits Conf. (Jeju, Korea)* pp 336–9
[19] Feng X L, White C J, Hajimiri A and Roukes M L 2008 *Nat. Nanotechnology* **3** 342–6
[20] Jackson J C, Windmill J F, Pook V G and Robert D 2009 *Proc. Natl. Acad. Sci. USA* **106** 10177–82
[21] Zlotnik A and Li J S 2014 arXiv:1401.1863
[22] Moehlis J, Shea-Brown E and Rabitz H 2006 *ASME J. Comput. Nonlinear Dyn.* **1** 358–67
[23] Nabi A and Moehlis J 2012 *J. Math. Biol.* **64** 981–1004
[24] Kirk D E 1970 *Optimal Control Theory: An Introduction* (Englewood Cliffs, NJ: Prentice-Hall)
[25] Tanaka H 2014 Optimal entrainment with smooth, pulse and square signals in weakly forced nonlinear oscillators *Physica* D: *Nonlinear Phenomena* at press
[26] Rudin W 1987 *Real and Complex Analysis* 3rd edn (New York: McGraw-Hill)
[27] Tanaka H A, Hasegawa A, Mizuno H and Endoh T 2002 *IEEE Trans. Circuits Syst. I* **49** 1271–8
[28] Nagashima T, Wei X, Tanaka H A and Sekiya H 2014 *IEEE Trans. Circuits Syst. I* doi:10.1109/TCSI.2014.2327276

[29] Matheny M H, Grau M, Villanueva L G, Karabalin R B, Cross M C and Roukes M L 2014 *Phys. Rev. Lett.* **112** 014101
[30] Yoshimura K and Arai K 2008 *Phys. Rev. Lett.* **101** 154101
[31] Goldobin D S, Teramae J N, Nakao H and Ermentrout G B 2010 *Phys. Rev. Lett.* **105** 154101
[32] Magnus J R and Neudecker H 1999 *Matrix Differential Calculus with Applications in Statistics and Econometrics* revised edn (Chichester: Wiley)